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EXTREME POINT SOLUTIONS

IN MATHEMATICAL PROGRAMMING:

AN OPPOSITE SIGN ALGORITHM

by

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Several important and efficient methods of solution of specific types of linear programming problems, such as the "flooding" or "flow" methods of Boldyreff and Ford-Fulkerson for network flows and the "decomposition" principle or "mixing routines" of Dantzig-Wolfe or Charnes-Cooper, have the distressing feature for managerial applications of sometimes providing optimal solutions which are not extreme point (or basic) solutions. Thereby, any immediate and automatic availability of the optimal dual evaluators of the original problem is not forthcoming. Indeed, several such experiences have resulted in abandonment of these techniques for managerial studies in certain industrial firms.

Aside from sensitivity analysis, it is also often desirable to start from an operable ("feasible") solution, not necessarily optimal, which is either suggested by company personnel or from various pertinent qualitative considerations, and to be able to proceed automatically to an extreme point (= basic) solution which is at least as good as the suggested one. [1] (Thereafter any one of the various methods could be employed to achieve optimality.)

The purpose of this paper is to exhibit how part of the technique of proof of the opposite sign theorem [2] can be employed in a simple algorithmic manner to achieve this end. We present an ALGOL code for executing this algorithm in a manner compatible (as a procedure) with standard programs.

Consider for the moment the linear programming problem written in the form:

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[1] There have been constructive methods which reduce any feasible solution to a basic solution, but which have not considered this important feature, or other features such as ability to employ a known but not necessarily feasible basis.

[2] See reference [3] on semi-infinite programming for a proof of it in general form.

$$\max \, f(\lambda) = \sum_{j=1}^{n} c_j \lambda_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} P_j \lambda_j = P_o$$

$$\lambda \geq 0$$

where P_j, P_o are all m-vectors. Let $\Lambda = \left\{ \lambda : \sum_{j=1}^{n} P_j \lambda_j = P_o \, , \, \lambda_j \geq 0 \right\}$.

We may assume that the problem is regularized. [1] This means that the solution set is non-empty and bounded, so the Opposite Sign Theorem applies:

Λ is generated by its extreme points : $\Longleftrightarrow$: $\sum_{j=1}^{n} a_j P_j = 0$, not all

$a_j = 0$ $\Longrightarrow$ some a_r , a_s are of opposite sign.

Now suppose some collection $B = \left\{ v_1, v_2, \ldots, v_q , \, q \leq m \right\}$ is a basis for the subspace of R^m spanned by the P_j. (Some or all of the v_i and P_j may be identical; it is not required that B be composed only of certain P_j's.) For notational convenience, rename any v_i which is not present in the collection $\left\{ P_j, j=1, \ldots, n \right\}$ as P_{n+i}, and let $I = \left\{ i : P_i \in B \right\}$.

Let B' denote a left inverse of the m $\times$ q basis matrix B; [2] then $a^T = B^{\#} P_k$ is the expression of P_k relative to B, or

$$\sum_{i \in I} a_i P_i = P_k \, ,$$

so that $\sum_{i \in I} a_i P_i + a_k P_k = 0$, where $a_k = -1$.

Now P_k can replace any P_i in the basis with $a_i \neq 0$, and by the opposite sign theorem some a_r and a_s are of opposite sign.

Let λ be _any_ solution to $P\lambda = P_o$, $\lambda \geq 0$. Assume that some vector $P_k \notin B$ occurs with positive λ-component in the expression of P_o; if no such vector exists then we already have an extreme point solution.

We now must consider the two possibilities which arise relative to the values of the corresponding a_j and λ_j components:

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[1] See Charnes-Cooper, reference [2], p. 424.

[2] See [4]; note that the matrix B need not be square.

Case 1.   There exists some $a_i \neq 0$ with corresponding $\lambda_i = 0$.

Choose one of these, say a_r , to designate a "remove" vector. No change will be effected in the functional to be considered until we have only P_j's in the basis; until this occurs we need not be concerned with the opposite sign property but are only choosing a basis from among the P_j vectors.

Case 2.   For every $a_i \neq 0$, $\lambda_i > 0$.

Form the quantities $\rho_1 = \min_{\substack{\lambda_j > 0 \\ a_j > 0}} \frac{\lambda_j}{a_j}$ and $\rho_2 = \min_{\substack{\lambda_j > 0 \\ a_j < 0}} \frac{\lambda_j}{|a_j|}$,

whose existence is guaranteed by the opposite sign theorem.   Then $\lambda^1 = \lambda - \rho_1 a$, $\lambda^2 = \lambda + \rho_2 a$ will have every component ≥ 0 and each will have at least one more zero component than λ as long as the number of $\lambda_j > 0$ is greater than q. Note also that because λ is a solution it is true that $\lambda + \rho a$ is also a solution for any real ρ , since $\sum_j a_j P_j = 0$, so that in particular λ^1 and λ^2 are both feasible solutions.

Now let $f(\lambda)$ be convex , i.e., for any λ^1 , λ^2

$$f((1-\gamma)\lambda^1 + \gamma\lambda^2) \leq (1-\gamma)f(\lambda^1) + \gamma f(\lambda^2) , \quad 0 \leq \gamma \leq 1 \text{ holds.}$$

Since $\lambda = \frac{\rho_2}{\rho_1 + \rho_2} \lambda^1 + \frac{\rho_1}{\rho_1 + \rho_2} \lambda^2$, it follows that

$$f(\lambda) \leq \frac{\rho_2}{\rho_1 + \rho_2} f(\lambda^1) + \frac{\rho_1}{\rho_1 + \rho_2} f(\lambda^2) , \text{ so that } \underline{not\ both}$$

$f(\lambda^1) < f(\lambda)$ and $f(\lambda^2) < f(\lambda)$; i.e., one of λ^1 , λ^2 yields at least as great a functional value as λ. Thus, the j which yields the minimum in the definition of ρ_1 , when $f(\lambda^1) \geq f(\lambda^2)$ (or of ρ_2 when $f(\lambda^2) \geq f(\lambda^1)$) serves to designate a vector P_r which is to be replaced by P_k in the basis. At this point we begin again with the new basis and new feasible λ as before.

Thus at each stage we remove one or more vectors from the original set, while maintaining at least as great a value as in the previous stage, until the original set is reduced to a set of linearly independent vectors, i.e., the corresponding λ is an extreme point with at least as great a functional value as the original given solution. (A corresponding result holds when a

concave functional is be be minimized.) We can summarize the above discussion in the following algorithm:

Step 1. Start with a feasible solution λ , $\sum_{j=1}^{n} P_j \lambda_j = P_o$, $\lambda \geq 0$, and a basis $B = \{P_i \ , \ i\epsilon I\}$.

Step 2. Find any $\lambda_k > 0$ with $P_k \not\in B$ to designate a vector to enter the basis; if none is available, λ is an extreme point and the process terminates.

Step 3. Obtain the α_i in $\sum_{i\epsilon I} \alpha_i P_i + \alpha_k P_k = 0$, where $\alpha_k = -1$, by computing $\alpha^T = B^{\#} P_k$.

Step 4. (i) If there exist $\alpha_i \neq 0$ with corresponding $\lambda_i = 0$, choose one, say α_r , to designate a remove vector P_r . Replace P_r by P_k in B, compute the new $B^{\#}$, and go to step 2.

(ii) Otherwise set $\rho_1 = \min\limits_{\substack{\alpha_i > 0 \\ \lambda_i > 0}} \dfrac{\lambda_i}{\alpha_i}$ and $\rho_2 = \min\limits_{\substack{\alpha_i < 0 \\ \lambda_i > 0}} \dfrac{\lambda_i}{|\alpha_i|}$.

Step 5. Let $\lambda^{(1)} = \lambda - \rho_1 \alpha$ and $\lambda^{(2)} = \lambda + \rho_2 \alpha$. Let r_1 , r_2 be a pair of indices for which ρ_1 , ρ_2 achieve their respective minima. Choose P_{r_1} or P_{r_2} to be removed ($= P_r$) according as $f(\lambda^{(1)})$ or $f(\lambda^{(2)})$ is the larger.

Step 6. Substitute P_k for P_r in the basis, compute the new $B^{\#}$, and return to step 1 with $\lambda^{(i)}$ as a new λ .

Observe that the algorithm terminates when a λ is reached such that each P_k with $\lambda_k > 0$ is in the current basis.

Notice again that it is possible to choose an initial basis consisting entirely of artificial or slack vectors (even though they may not be part of the original set of vectors), and that the procedure will automatically find a basic solution in terms of the original vectors.

It should be expressly noted that the process of choosing a vector to enter the basis (any vector with $\lambda_j > 0$, and not already in the basis, may be chosen), computing a column of α_j ("substitution ratios"), selecting a

vector to be removed from the basis (the one at which either the ρ_1 or the ρ_2 minimum ratio was achieved), and transforming the matrix of the basis inverse to reflect this change of basis corresponds closely in structure with that of the modified simplex method of A. Charnes and C. E. Lemke, [1]/ although the actual criteria governing these operations are different. In particular, just as in the modified simplex method, the possibility of an unbounded solution is indicated by the absence of a positive α_i (pivot element), although, since there are normally two candidates for removal from the basis, this condition is critical only if the other candidate would cause a worsening of the solution; e.g., if we are maximizing, the availability of an infinite minimum will not cause an error stop; we would simply remove the vector associated with the minimum ρ_2 ratio from the basis if no decrease in the functional is caused thereby.

It is conceivable that the initial "solution" presented to the procedure could be really not a solution at all. If λ is the "solution" presented and $P\lambda = \bar{P} \neq P_o$, the resulting "basic solution" will be a basic solution to $P\lambda = \bar{P}$, since the algorithm never refers to the actual stipulations vector P_o ! For production use, therefore, it would be worth while to check before using the procedure that the solution proposed really is feasible.

The ALGOL procedure presented below assumes that $f(\lambda)$ is linear, i.e., $f(\lambda) = c^T\lambda$, which is both concave and convex. The roles of the formal parameters used in the procedure declaration are described in the following table.

1/   See [5] .

Formal Parameter	Description
m	The number of constraints. (The number of rows in P.)
n	The number of columns in P, exclusive of P_o.
size	The number of vectors in the basis, which is the dimension of the space spanned by the P_j. In most cases this will be equal to m.
basis	The list of indices of the P_i in the starting basis; if no starting basis is supplied then the elements of this array are irrelevant when the procedure is called.
matrix	The m x n constraint matrix.
inverse	A size x m (left) inverse of the initial basis; again, the elements are irrelevant if no basis is specified. One additional row should be adjoined which will contain the dual evaluators.
lambda	The initial feasible solution, whose components must be in the same order as the columns of the coefficient matrix.
c	The components of the objective, in the same order as those of lambda.
objective	An integer which should be zero or one if minimization or maximization, respectively, is desired.
option	An indicator which should be set to one if no starting basis is specified in which case the m x m identity matrix will be used, so that size must be equal to m; if a basis and inverse are provided, this indicator should be set to 2.
unbounded	The name of a sentinel which will normally be set to zero but which assumes a non-zero value in case an infinite optimum is attainable.

In terms of the ALGOL identifiers used as formal parameters of the procedure, upon completion of the process the integer array, basis , contains the numbers of the vectors in the basic solution; the array, lambda , contains the actual values of the λ_j , and unbounded is equal to 0; if not zero, it was set to the number of the vector attempting to enter the basis when unboundedness was noticed and the process terminated. The basis (left) inverse is contained in inverse , which also includes a size + 1st row containing the dual evaluators for the current basic solution.

ALGOL Procedure

```
procedure purify (m, n, size, basis, matrix, inverse, lambda, c, objective,
                  option, unbounded) ;

value m, n, size, objective, option ;

integer m, n, size, objective, option, unbounded ;

real array matrix, inverse, lambda, c ;

integer array basis ;

begin integer i, j, k, r, r1, r2 ;

        array alpha [1 : size + 1] ;

        real rho1, rho2, t, d1, d2 ;

        unbounded : = 0 ;

        comment if necessary, generate augmented identity matrix  for the
                left inverse and dual evaluators ;

        if option = 1 then for j : = 1 step 1 until n do

        begin for i : = 1 step 1 until m + 1 do

                inverse [i,j] : = if i = j then 1 else 0 ;

                basis [j] : = n+j

        end

        look for some lambda greater than zero to determine a vector
        to enter the basis ;
```

```
again : for j: =1 step 1 until n do

        if lambda [j] <10-7 then lambda [j] : = 0

        else begin for i : =1 step 1 until size do

                        if basis [i] =j then goto skip it ;

                        comment if we get to here, vector j was
                                not already in the basis ;

                        k: =j ;

                        goto come in ;

skip it:          end

        if we get to here, no vector k can be found with lambda [k] > 0 and
        which is not already in the basis, hence we are done. ;

        goto finished ;

come in: for i:=1 step 1 until size + 1 do

        begin alpha [i] : = 0 ;

              for j: = 1 step 1 until m do

              alpha [i] : = alpha [i] + inverse [i,j]* matrix [j,k]

        end ;

        alpha [size + 1] : = alpha [size + 1] - c[k] ;

        rho1  : = rho2  : = 10 20 ;

        comment now find two candidates for removal ;

        for i : = 1 step 1 until size do

        if alpha [i] >  10 -7 then

                begin if basis [i] > n then

                        begin r : = i ; goto transform end

                        else if lambda [basis [i] ] < 10 -7 then

                        begin r : = i ; goto transform end
```

```
                        else begin t : = lambda [basis [i] ] / alpha [i] ;

                                    if t < rho1   then

                                    begin rho1  : = t ; r1 : = i end

                        end

            end

else if alpha [i] < -10 -7 then

        begin if basis [i] > n then

                    begin r : = i ; goto transform end

              else if lambda [basis [i] ] < 10 -7 then

                    begin r : = i ; goto transform end

              else begin t : = - lambda [basis [i] ] / alpha [i] ;

                            if t < rho2   then

                            begin rho2  : = t ; r2 : = i end

                    end

        end ;

if lambda [k] < rho2   then

            begin rho2  : = lambda [k] ; r2 : = -1 end ;

d1 : = c[k] * rho1  ;

d2 : = -c[k] * rho2  ;

for i : = 1 step 1 until size do

begin t := c [basis [i] ] * alpha [i] ;

        d1 : = d1 - rho1  * t ;

        d2 : = d2 + rho2  * t

end ;

comment now choose the best vector to be removed, either r1 or r2 ;
```

```
        if objective = 0 ≝ dl ≤ d2

        then goto takeout 1 else goto takeout 2 ;

takeout 1:  if rhol = 10 20 then begin unbounded : = k ; goto finished end ;

        for i : = 1 step 1 until size do

                lambda ⌊basis [i] ] : = lambda ⌊basis [i] ] - rhol  * alpha [i];

        lambda [k] : = lambda ⌊k] + rhol   ;

        r : = r 1 ;

        goto transform ;

takeout 2:  for i : = 1 step 1 until size do

                lambda [basis [i] ] : = lambda [basis [i] ] + rho2  * alpha [i];

         lambda ⌊k] : = lambda ⌊k] - rho2 ;

        if r2 = -1 then goto again else r : = r2 ;

transform:  for j : = 1 step 1 until m do

        begin inverse ⌊r,j] : = inverse [r,j] / alpha [r] ;

                for i : = 1 step 1 until size + 1 do

                if i ≠ r then

                        inverse ⌊i,j] : = inverse ⌊i,j] - inverse ⌊r,j] * alpha ⌊j]

        end ;

        basis [r] : = k ;

        goto again ;

finished:  end ;
```

Example

We present a simple example with a summary of the iterations taken to reach a basic solution, starting from an initial basis not consisting of vectors taken from the original coefficient matrix.

$$\text{max} \quad 2x_1 + x_2 + 4x_3 + 3x_4 + x_5$$

$$\text{subject to} \quad 5x_1 + x_2 + 6x_3 \qquad + 3x_5 = 26$$

$$7x_1 + x_2 + 2x_3 - x_4 - 2x_5 = 5$$

$$\text{and all } x_i \geq 0$$

Stage	Come-in Vector	Remove Vector	Current Basis	λ	$f(\lambda)$	Case
Initial	--	--	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(2,1,1,6,3,0,0)$	30	--
1	P_1	P_6	$\begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$	$(2,1,1,6,3,0,0)$	30	Case 1
2	P_2	P_7	$\begin{pmatrix} 5 & 1 \\ 7 & 1 \end{pmatrix}$	$(2,1,1,6,3,0,0)$	30	Case 1
3	P_3	P_1	$\begin{pmatrix} 1 & 6 \\ 1 & 2 \end{pmatrix}$	$(0,17,0,6,3,0,0)$	38	Case 2
4	P_4	P_3	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$(0,17,0,6,3,0,0)$	38	Case 1
5	P_5	P_5	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$(0,26,0,21,0,0,0)$	89	Case 2

The optimum value for this problem is 104 - 3/5 .

R E F E R E N C E S

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13. ABSTRACT

Several important and efficient methods of solution of specific types of linear programming problems have the feature of sometimes providing optimal solutions which are not extreme-point (or basic) solutions, so that important and useful analyses provided by knowledge of the optimal dual evaluators are not available. It is also often desirable to be able to begin with a solution suggested by knowledgeable persons with experience in the field (or other considerations) and to proceed immediately to a basic solution at least as good as the suggested one.

In this paper it is shown how part of the technique of proof of the opposite sign theorem can be employed in a simple algorithm to achieve this end. This method is equally valid when maximizing a nonlinear but convex objective function. A tested ALGOL code is provided for executing the algorithm in a manner compatible (as a procedure) with other programs.

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